Diploma Seminar

Presentation Subject:

Numerical and optimization methods

University:

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Topics

- Introduction
- Formulation of an optimization problem
- Linear Programming
- Nonlinear constrained optimization
- Example of Formulation of an optimization problem
- Unconstrained minimization techniques
- Gradient method
- Termination of an iterative minimization algorithm
- Newton method
- Penalty method

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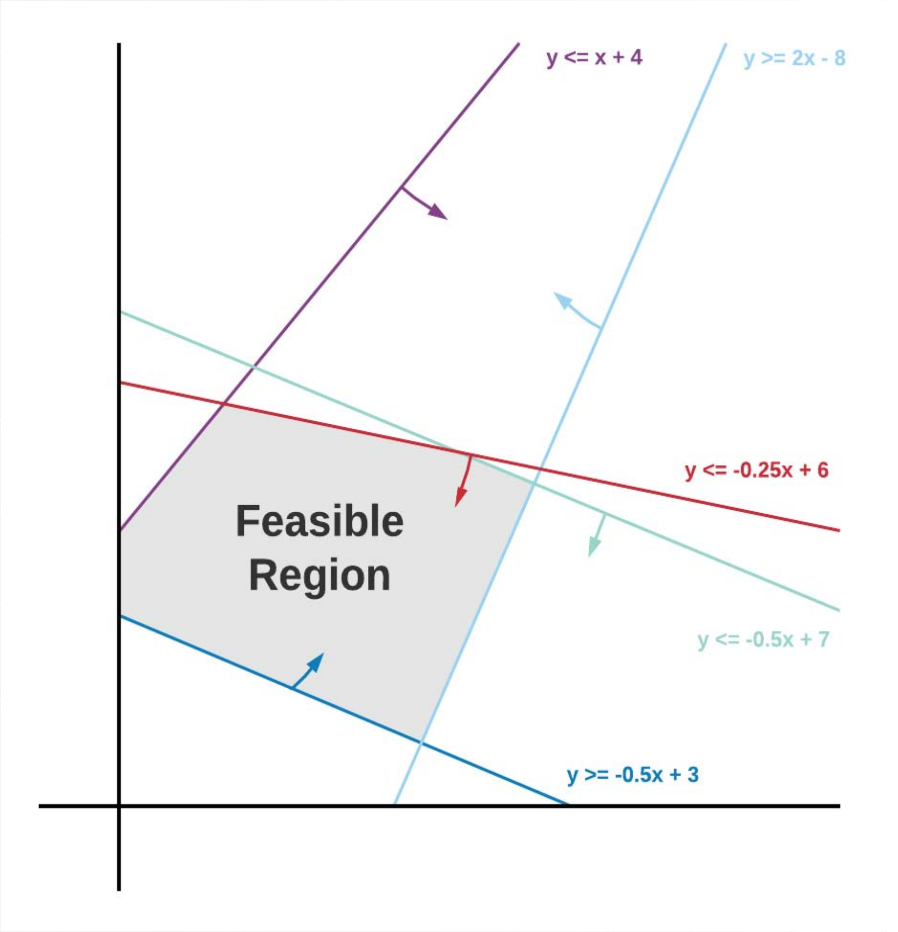
Introduction

- Optimization problem is strongly connected with decision making.
- Decision making means to choose among various possibilities.
- Generally, there are constraints (requirements) which do not allow to make any decision. Decision which conforms to such requirements is called constrained (feasible) decision
- There is an overall goal or objective for the process to be optimized. According to that goal the decisions can be called better or worse. Decision criterion.

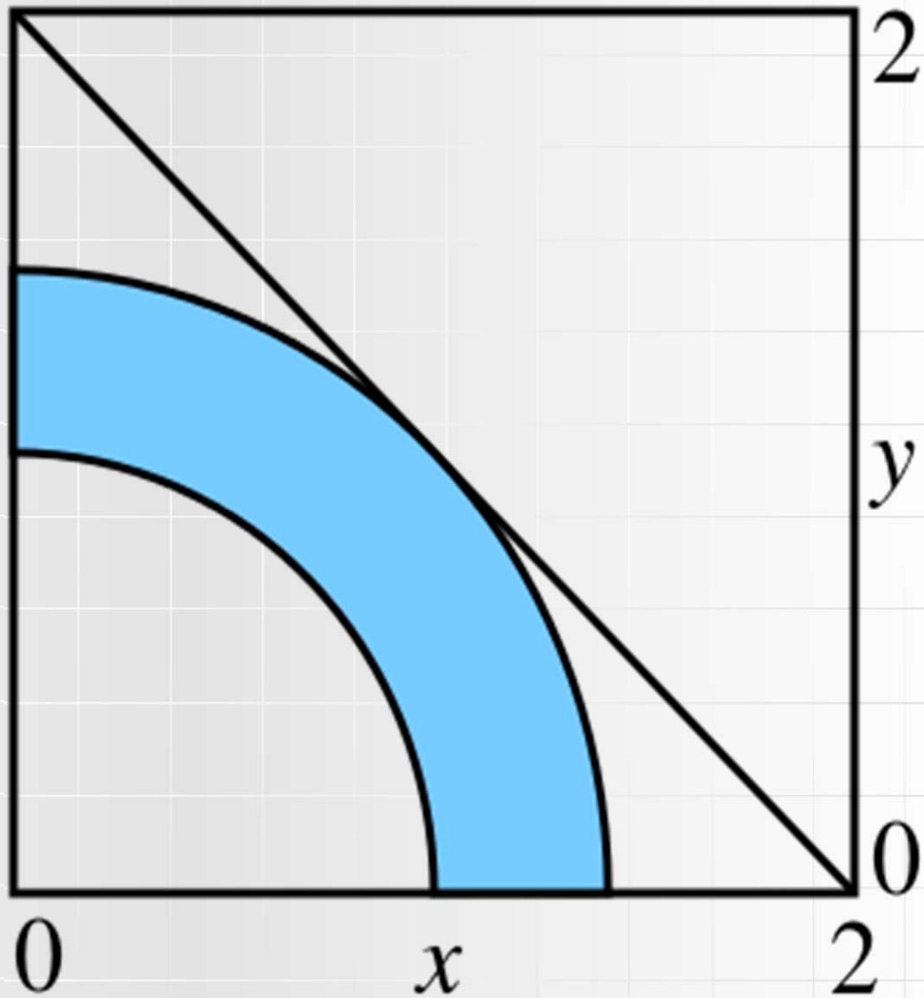
Linear Programming

- Linear programming is a mathematical concept used to determine the solution to a linear problem. Typically, the goal of linear programming is to maximize or minimize specified objectives, such as profit or cost.
- Linear functions such as objectives and constraints are displayed as straight lines on a graph
- The optimal value for an objective will often be in a corner of this feasible region, as this will be the maximal or minimal feasible value for the objective.



Nonlinear constrained optimization

- Nonlinear optimization is the process of solving optimization problems that concern some of the nonlinear constraints or nonlinear objective functions. It involves minimizing or maximizing a nonlinear objective function subject to bound constraints, nonlinear constraints, etc.
- These constraints can be inequalities or equalities.
- Constrained nonlinear programming involves finding a vector x that minimizes a nonlinear function f(x) subjected to one or more constraints.



Formulation of an optimization problem

- Precisely, the formulation of an optimization problem involves:

1) Selecting one or more optimization variables (choices)
2) Identifying a set of constrains (mostly in the form of a set of equations or inequations)
3) Choosing an objective function (criteria for “best decision”)

Example of formulation of optimization problem

- There are A and B products.

  For one A product we need 5 units of raw material S1 and 7 units of raw material S2.

  For one B product we need 4 units of raw material S1 and 10 units of raw material S2 .

Every day 200 units of S1 and 300 units of S2 are delivered.

Product A is sold at 50 € per exemplar and B at 60 € per exemplar. Production costs are 10 € per exemplar. We do not differentiate raw material costs (simplification)

Example of formulation of optimization problem(cont)

- Optimization variables

x1 – daily production of A product.

x2 – daily production of B product.

- Constraints

$5x_1 + 4x_2 \leq 200$ and $7x_1 + 10x_2 \leq 300$

- Optimization functions (maximizing profit)

Profit = Sell Price – cost

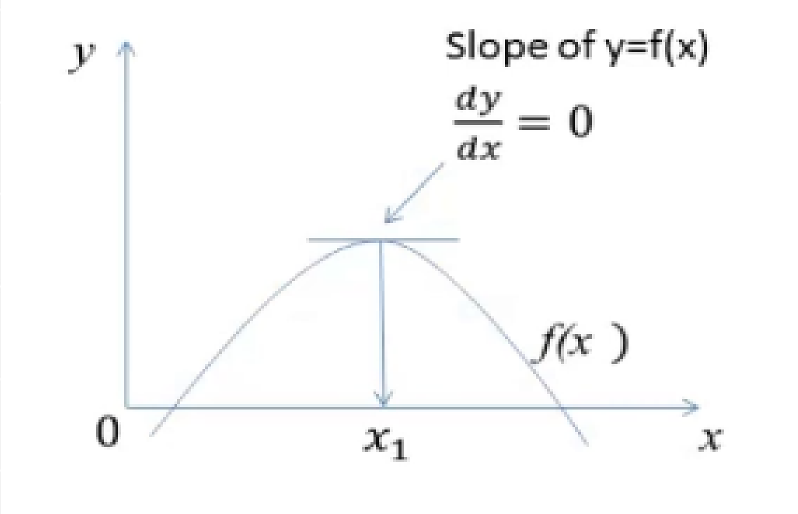
$F(x_1, x_2) = 50x_1 + 60x_2 - 10(x_1 + x_2) = 40x_1 + 50x_2$

Unconstrained minimization techniques

- Let y = f(x)

Maximization of f(x) requires

i) $\frac{dy}{dx} = 0$ ii) $\frac{d^2y}{dx^2} < 0$

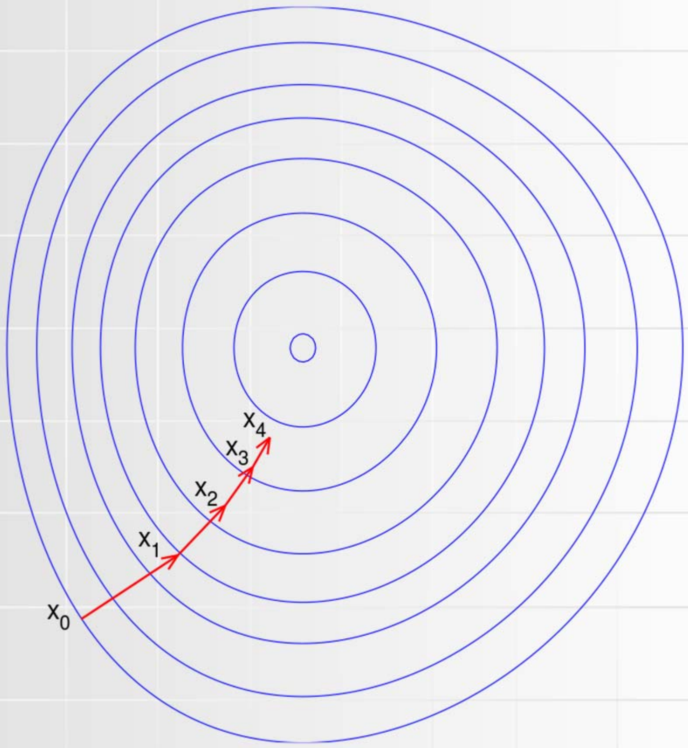


Minimization of f(x) requires

i) $\frac{dy}{dx} = 0$ ii) $\frac{d^2y}{dx^2} > 0$

Gradient method

- gradient descent (also often called steepest descent) is a first-order iterative optimization algorithm for finding a local minimum of a differentiable function.
- The idea is to take repeated steps in the opposite direction of the gradient (or approximate gradient) of the function at the current point, because this is the direction of steepest descent. Conversely, stepping in the direction of the gradient will lead to a local maximum of that function; the procedure is then known as gradient ascent.



Gradient method

- we have to compute the descent direction and find the minimum in that direction.
- First of all, we have to define the starting point x^0. Further, as said in the previous point: find the descent direction d^k of the step and seek the best value of the optimization problem for x^k in that direction
- 3 steps to solve the problem

1) Find the decent direction d^k
2) Find the minimum of F in the direction d^k , that is Find the value of α^k minimizing $\Phi(\alpha)$ =F(x^k+ αd^k) where(α-step length)
3) Find the starting point for next iteration step $x^{k+1} = x^k + \alpha d^k$

Gradient method (cont)

- Advantages

- It is characterized by simplicity

- Works easily with larger models

- Disadvantages

- It may take more iterations than other models so it is not the most efficient

$$\mathbf{x}^{k+1} = \mathbf{x}^{k} - \alpha^{k}\nabla F\left(\mathbf{x}^{k}\right)$$

$$x_1^{k+1} = x_1^{k} - \alpha^{k}\left(\frac{\partial F}{\partial x_1}\right)_{x_1 = x_1^{k}}$$

$$x_2^{k+1} = x_2^{k} - \alpha^{k}\left(\frac{\partial F}{\partial x_2}\right)_{x_2 = x_2^{k}}$$

$$\vdots$$

$$x_n^{k+1} = x_n^{k} - \alpha^{k}\left(\frac{\partial F}{\partial x_n}\right)_{x_n = x_n^{k}}$$

Criteria for the termination of an iterative minimization algorithm

- When should we stop the iterative algorithm?
  It is possible, that for some k we obtain $\nabla f\ (x^k)$ = 0 which is the necessary condition for a minimum. We can use the point to stop the algorithm.
- Practically, doing numerical computation we very seldom have a convenient situation, that for given k $\nabla f\ (x^k)$ = 0 ,Instead, we check the value of the norm $\|\nabla f\ (x^k)\|$ Assuming, that x^k is a sufficient approximation of x if the norm is smaller than a chosen value (epsilon ε) ($\|x^{k+1} - x^k\|$ < ε)

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Newton’s Method

- The method of steepest descent uses only first derivatives (gradients) in selecting the search direction. This is not the most effective strategy.
- If higher derivatives are used, the resulting iterative algorithm may perform better than the steepest decent method
- Newton’s Algorithm (also called Newton-Raphson Method) uses first and second derivatives and indeed does perform better then the steepest decent method if the initial point is close to the minimizer.

Penalty methods

- The penalty method replaces a constrained optimization problem by a series of unconstrained problems whose solutions must converge to the solution of the original constrained problem
- The corresponding minimization problems are formed by adding a penalty term to the objective function.
- The penalty term grows when the constrains are violated and is 0 in the region where constrains are not violated. Or it makes it impossible to go outside the region of feasible points
- The penalty term is usually a product of possible penalty coefficient and a penalty function

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Thanks for your attention !